Librations of Gravity-Oriented Satellites in Elliptic Orbits through Atmosphere

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The coupled librational dynamics of gravity-oriented, axisymmetric satellites in elliptic orbits is investigated. An approximate closed form solution is obtained using variation of parameter approach to study the effect of eccentricity and inertia on the response. The plots for stability in the large, generated numerically, show the transverse motion to be relatively more stable. Since in the presence of atmosphere the static equilibrium configuration varies continuously with the satellite's orbital position, the response becomes relatively complex and the stability region diminishes rapidly. However, with a suitable choice of a velocity-sensitive, semipassive controller, the normally destabilizing aerodynamic moment can be used to advantage in damping the librational motion.

Nomenclature‡

a,b = amplitude of librations in and across orbital plane, respectively
e = eccentricity of orbit
f,g = nonlinear functions, Eq. (7)
f^*,g^* = approximation to f and g, respectively
1_m = moment arm, Eq. (26)
m = mass of satellite
n_1,n_2 = frequencies of librations in and across orbital plane, respectively
v = orbital velocity
x,y,z = principal body coordinates with z along the axis of symmetry
A_f = flap area
B_f,B_{fE},B_{fp} = aerodynamic coefficient in circular orbit, elliptic orbit and at perigee, respectively
C_1 = ratio of transverse to axial cross-sectional areas of satellite, $\pi D_0/4L_0$
C_D,C_L = drag and lift coefficients, respectively
D_0,L_0 = cylindrical satellite's diameter and length, respcetively
I_{xx},I_{yy},I_{zz} = moments of inertia about x,y,z axes, respectively
K_i = inertia parameter, $(I - I_{zz})/I$, $I = I_{xx} = I_{yy} > I_{zz}$
R = radius of the Earth
S_i,C_i,T_i = sin(i), cos(i), tan(i), respectively
T = kinetic energy
U_a,U_g = aerodynamic and gravitational potential, respectively
β_1,β_2 = phase angles, Eq. (8)
ζ = $n_1\theta + \beta_1$
η = $n_2\theta + \beta_2$
θ = angular position of satellite as measured from pericenter
ψ,ϕ,λ = Eulerian angles corresponding to pitch, roll and yaw, respectively
μ = gravitational constant
μ_i = proportionality constant in controller
ρ = atmospheric density
$\tau_{1\max},\tau_{2\max}$ = nondimensionalized maximum stabilizing torques in ψ and ϕ degrees of freedom, respectively

Subscripts

e = value of parameter at stable equilibrium configuration
$0,o$ = initial condition
p = value of parameter at pericenter

Presented as Paper 71-89 at the AIAA 9th Aerospace Sciences Meeting, New York, January 25–27, 1971; submitted February 16, 1971; revision received July 26, 1971. The research for this paper was supported, in part, by the Defense Research Board of Canada, Grant 9551-18.
Index category: Spacecraft Attitude Dynamics and Control.
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‡ Dots and primes indicate differentiation with respect to time t and θ, respectively.

Introduction

THE attitude stabilization using gravity gradient torque has been a subject of extensive investigation.[1–5] A bulk of the literature is limited to the analysis of the planar librations, probably because of the nonlinear, nonautonomous and coupled nature of the equations of motion. However, as pointed by Kane[6] and further substantiated by Breakwell and Pringle,[7] motion across the orbital plane cannot be ignored, especially for large amplitude librations. Apparently some simplification of the problem can be achieved by restricting the analysis to axisymmetric satellites in circular orbits.

Modi and Brereton[8] have analyzed the stability bounds of dumbbell shaped satellites, executing coupled librations in circular orbits, numerically. Extension of the analysis to an arbitrary axisymmetric satellite through an approximate analytical technique was presented by Modi and Shrivastava.[9] More recently, the authors investigated,[10] numerically, the effects of inertia on the regions of possible motion, response characteristics and stability bounds of the system. The concept of integral manifolds[11] was used for a concise presentation of the nature of the solutions.

An analysis of the preceding autonomous system in presence of aerodynamic moment was presented by Shrivastava and Modi.[12] The stable equilibrium orientation, found using infinitesimal technique and Liapunov's direct method, emphasized the effect of the environment. The plots of allowable impulsive disturbances, which a satellite at equilibrium can sustain without tumbling, showed space vehicles with large inertias to be relatively more stable at higher altitudes. However, shorter satellites exhibited better stability characteristics in presence of a large aerodynamic moment.

Studies of the nonautonomous gravity oriented systems even in absence of environmental forces, have been restricted to satellites negotiating near-circular orbits.[7] On the other hand, it may be recognized that metereological, earth resources, military reconnaissance, etc., satellites using close Earth orbits for better resolutions, can have their life span increased through use of elliptic trajectories.

This paper investigates coupled librational dynamics of such nonautonomous systems. In the beginning an approximate closed form analytical solution is obtained for the system in absence of aerodynamic moment using a modification of Butenin's[13] approach. This is followed by a numerical response and stability analysis in the large over a wide range of inertia parameters and eccentricities. Next the effect of aerodynamic moment on the equilibrium configuration, system response, and stability are studied in detail. A convenient condensation of the results, affected through plots showing allowable impulsive disturbances for various values of the system parameters, should prove useful during satellite design. Finally, a possibility of utilizing the normally destabilizing aerodynamic moment to advantage is explored through the use of a velocity-sensitive, semipassive controller.

Formulation of the Problem

Consider an arbitrary axisymmetric, gravity oriented, rigid satellite in an orbit about 0 executing coupled librations (Fig. 1). With respect to the principal body coordinates x,y,z, the expression for the potential and kinetic energies to $0(1/r^3)$, in absence of the atmosphere, can be written as[8,9]:

$$U_g = -\mu m/r + \mu(I - I_{zz})(1 - 3C_\psi^2 C_\phi^2)/2r^3 \quad (1)$$

$$T = m(\dot{r}^2 + r^2\dot{\theta}^2)/2 + I[\dot{\phi}^2 + (\dot{\theta} + \dot{\psi})^2 C_\phi^2]/2 + I_{zz}[\dot{\lambda} - (\dot{\theta} + \dot{\psi})S_\phi]^2/2 \quad (2)$$

Since λ does not appear explicitly in the expression for Lagrangian, the conjugate moment p_λ must be a constant of the motion, i.e.,

$$p_\lambda = \partial(T - U_g)/\partial\dot{\lambda} = I_{zz}[\dot{\lambda} - (\dot{\theta} + \dot{\psi})S_\phi] = \text{const} \quad (3)$$

For a nonspinning satellite the constant must be equal to zero, thus simplifying Eq. (2). Using the Lagrangian formulation, the governing equations of motion in the r, θ, ψ, and ϕ degrees of freedom can be written as

$$\ddot{r} + r\dot{\theta}^2 + \mu/r^2 - 3\mu(I - I_{zz})(1 - 3C_\psi^2 C_\phi^2)/2mr^4 = 0 \quad (4a)$$

$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} + I[(\ddot{\theta} + \ddot{\psi})C_\phi^2 - \dot{\phi}(\dot{\theta} + \dot{\psi})S_{2\phi}]/m = 0 \quad (4b)$$

$$\ddot{\psi} + \ddot{\theta} - 2\dot{\phi}(\dot{\theta} + \dot{\psi})T_\phi + 3\mu K_i S_\psi C_\psi/r^3 = 0 \quad (4c)$$

$$\ddot{\phi} + [(\dot{\theta} + \dot{\psi})^2 + 3\mu K_i C_\psi^2/r^3]S_\phi C_\phi = 0 \quad (4d)$$

The orbital perturbation due to the librational motion being small,[14] Eqs. (4a) and (4b) reduce to the classical Keplerian relations. Introducing the change in the independent variable from time t to θ, the equations of librational motion (4c) and (4d) transform to

$$\psi''(1 + eC_\theta) - 2eS_\theta(\psi' + 1) - 2\phi'(\psi' + 1)(1 + eC_\theta)T_\phi + 3K_i S_\psi C_\psi = 0 \quad (5a)$$

$$\phi''(1 + eC_\theta) - 2eS_\theta\phi' + [(1 + \psi')^2(1 + eC_\theta) + 3K_i C_\psi^2]S_\phi C_\phi = 0 \quad (5b)$$

These second-order, coupled, nonlinear, nonautonomous equations of motion remain invariant under the transformation: θ,ψ,ϕ to $\theta,\psi,-\phi$; $-\theta,-\psi,\phi$ or $-\theta,-\psi,-\phi$.

Approximate Solution and System Response

In absence of a known, exact, closed-form solution to such a complex system, it was decided to analyze the problem approximately using modification of Butenin's variation of parameter technique.

Replacing the trignometric functions of the dependent variables by their series, ignoring fifth and higher degree terms in ψ, ϕ, and their derivatives and collecting nonlinear terms and forcing function on the right hand side, Eqs. (5a) and (5b)

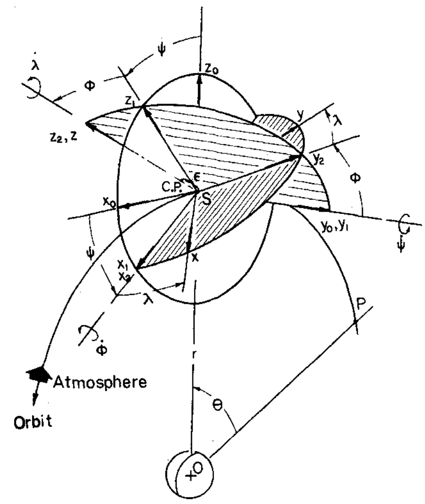


Fig. 1 Geometry of motion: 0 = center of force; S = center of mass; $C.P.$ = center of pressure; p = perigee.

take the form

$$\psi'' + 3K_i\psi \approx 2eS_\theta + 2[(eS_\theta\psi' + K_i\psi^3)/(1 + eC_\theta) + \phi'\phi(\psi' + 1) + \phi'\phi^3/3] \quad (6a)$$

$$\phi'' + (1 + 3K_i)\phi \approx \{[2eS_\theta\phi' + K_i\phi(2\phi^2 + 3\psi^2)]/(1 + eC_\theta) + 2\phi^3(1 - 2\psi')/3 - \psi'\phi(2 + \psi')\} \quad (6b)$$

or

$$\begin{aligned}\psi'' + n_1^2\psi &= 2eS_\theta + f(\psi,\psi',\phi,\phi',\theta)\\ \phi'' + n_2^2\phi &= g(\psi,\psi',\phi,\phi',\theta)\end{aligned} \quad (7)$$

The solution of the corresponding linear system, i.e., $f = g = 0$ is given as

$$\begin{aligned}\psi &= aS_{(n_1\theta+\beta_1)} + 2eS_\theta/(n_1^2 - 1)\\ \phi &= bS_{(n_2\theta+\beta_2)}\end{aligned} \quad (8)$$

where a, b, β_1 and β_2 are constants which can be determined from initial conditions. A solution in similar form is now sought allowing the amplitude and phase angles to be functions of θ, i.e.,

$$\psi = a(\theta)S_{[n_1\theta+\beta_1(\theta)]} + 2eS_\theta/(n_1^2 - 1) \quad (9a)$$

$$\phi = b(\theta)S_{[n_2\theta+\beta_2(\theta)]} \quad (9b)$$

a, b, β_1, and β_2 can be expressed as functions of θ plus a constant. Thus the solution in the present form involves eight unknowns, four of which can be determined by the initial conditions whereas the remaining have to be found through the imposition of constraints.

Keeping the first derivatives of Eqs. (9a) and (9b) similar to the homogeneous solution gives two of the constraint relations

$$a'S_\zeta + a\beta_1'C_\zeta = 0 \quad (10a)$$

$$b'S_\eta + b\beta_2'C_\eta = 0 \quad (10b)$$

Other two relations are obtained by substituting Eqs. (9a) and (9b) in the equations of motion (7) giving

$$a'n_1C_\zeta - an_1\beta_1'S_\zeta = f^* \quad (10c)$$

$$b'n_2C_\eta - bn_2\beta_2'S_\eta = g^* \quad (10d)$$

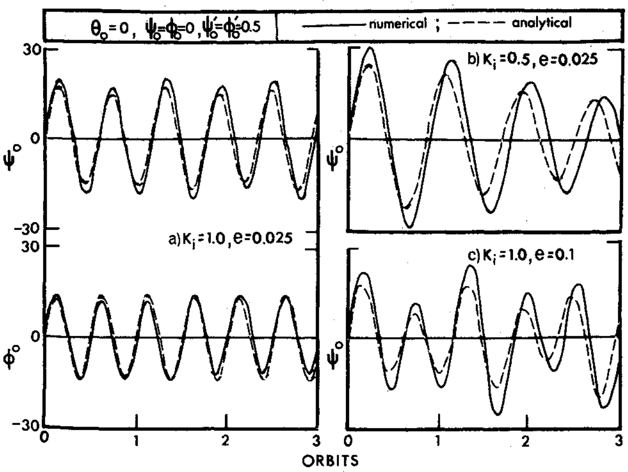


Fig. 2 Representative comparison of the responses, generated using Butenin's approach and numerical method, showing effects of satellite inertia and orbital eccentricity.

where

$$
\begin{aligned}
f^* &= f[aS_\xi + 2eS_\theta/(n_1^2 - 1),\ an_1C_\xi + \\
&\qquad 2eC_\theta/(n_1^2 - 1), bS_\eta, bn_2C_\eta, \theta] \\
g^* &= g[aS_\xi + 2eS_\theta/(n_1^2 - 1), an_1C_\xi + \\
&\qquad 2eC_\theta/(n_1^2 - 1), bS_\eta, bn_2C_\eta, \theta]
\end{aligned} \tag{11}
$$

Solving Eq. (10) yields

$$
\begin{aligned}
a' &= f^*C_\xi/n_1, \quad \beta_1' = -f^*S_\xi/an_1 \\
b' &= g^*C_\eta/n_2, \quad \beta_2' = -g^*S_\eta/an_2
\end{aligned} \tag{12}
$$

Since f^*, g^* are small for small disturbances, a, b, β_1, and β_2 are slowly varying parameters. Using their average values over a period gives

$$
a' = (1/8\pi^3 n_1) \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} f^*C_\xi d\xi d\eta d\theta, \text{ etc.} \tag{13}
$$

The solution, therefore, becomes

$$
\begin{aligned}
\psi &= aS_{[(3K_i)^{1/2}\theta+\beta_1]} + 2eS_\theta/(3K_i - 1) \\
\phi &= bS_{[(3K_i+1)^{1/2}\theta+\beta_2]}
\end{aligned} \tag{14}
$$

where

$$
\begin{aligned}
a &= \{\psi_0^2 + [\psi_0' - 2e/(3K_i - 1)]^2/3K_i\}^{1/2} \\
b &= [\phi_0^2 + \phi_0'^2/(1 + 3K_i)]^{1/2} \\
\beta_1 &= -(3K_i)^{1/2}a^2\theta/4 + \tan^{-1}\{(3K_i)^{1/2}\psi_0/ \\
&\qquad [\psi_0' - 2e/(3K_i - 1)]\} \\
\beta_2 &= -\{b^2[1 + 3K_i/(1 - e^2)^{1/2}] - 3a^2K_i[1 - 1/ \\
&\qquad (1 - e^2)^{1/2}] + 4e^2/(3K_i - 1)\}\theta/[4(1 + 3K_i)^{1/2}] + \\
&\qquad \tan^{-1}[(1 + 3K_i)^{1/2}\phi_0/\phi_0']
\end{aligned} \tag{15}
$$

To establish the accuracy of the analytical technique the equations of motion were integrated numerically using Adams-Bashforth predictor corrector quadrature with Runge-Kutta starter.[15] A step size of 3° gave the results of sufficient accuracy without involving excessive computational time.[16] The librational response as affected by satellite inertia, orbital eccentricity and external disturbance was obtained over fifty orbits. However, for conciseness, the comparison between the two methods is limited to initial regions in Fig. 2.

As in the case of a circular orbit,[9] the solution appears to agree well with the numerical results, particularly for the motion across the orbital plane, even for a disturbance of an appreciable magnitude, $\psi_0' = \phi_0' = 0.5$. The effect of eccentricity is reflected through motion modulations. Both methods show that a larger amplitude, smaller frequency motion, with a period of the order same as that of the orbit, is excited in the orbital plane when the satellite is subjected to identical disturbances in the two degrees of freedom (Fig. 2a). The phase discrepancy between the solutions appears to grow with time. In most cases the phase shift remains less than $\pi/2$ rad in 50 orbits. The librational amplitude predicted by the approach is, in general, smaller than the actual. Thus the resulting analytically obtained stability bound is likely to be larger.

Although, the agreement deteriorates in ψ with decreasing slenderness of the satellite (Fig. 2b) and increasing eccentricity (Fig. 2c), the analysis continues to predict the general behavior, at least qualitatively. Reduction of K_i or increase in e enhances the amplitude modulations, especially for the planar degree of freedom.

Both the solutions suggest that in the absence of any initial disturbance, appreciable oscillations in the orbital plane are excited due to eccentricity of the orbit. A presence of any cross motion is likely to induce small perturbations in the planar librations, however, the analytical approach fails to predict this phenomenon.

Since in the actual practice the gravity gradient satellites possess large K_i, normally move in circular or almost circular orbits, and exhibit moderate pointing accuracy, the analytical solution can be applied with confidence, at least for preliminary design purposes.

Stability Plots

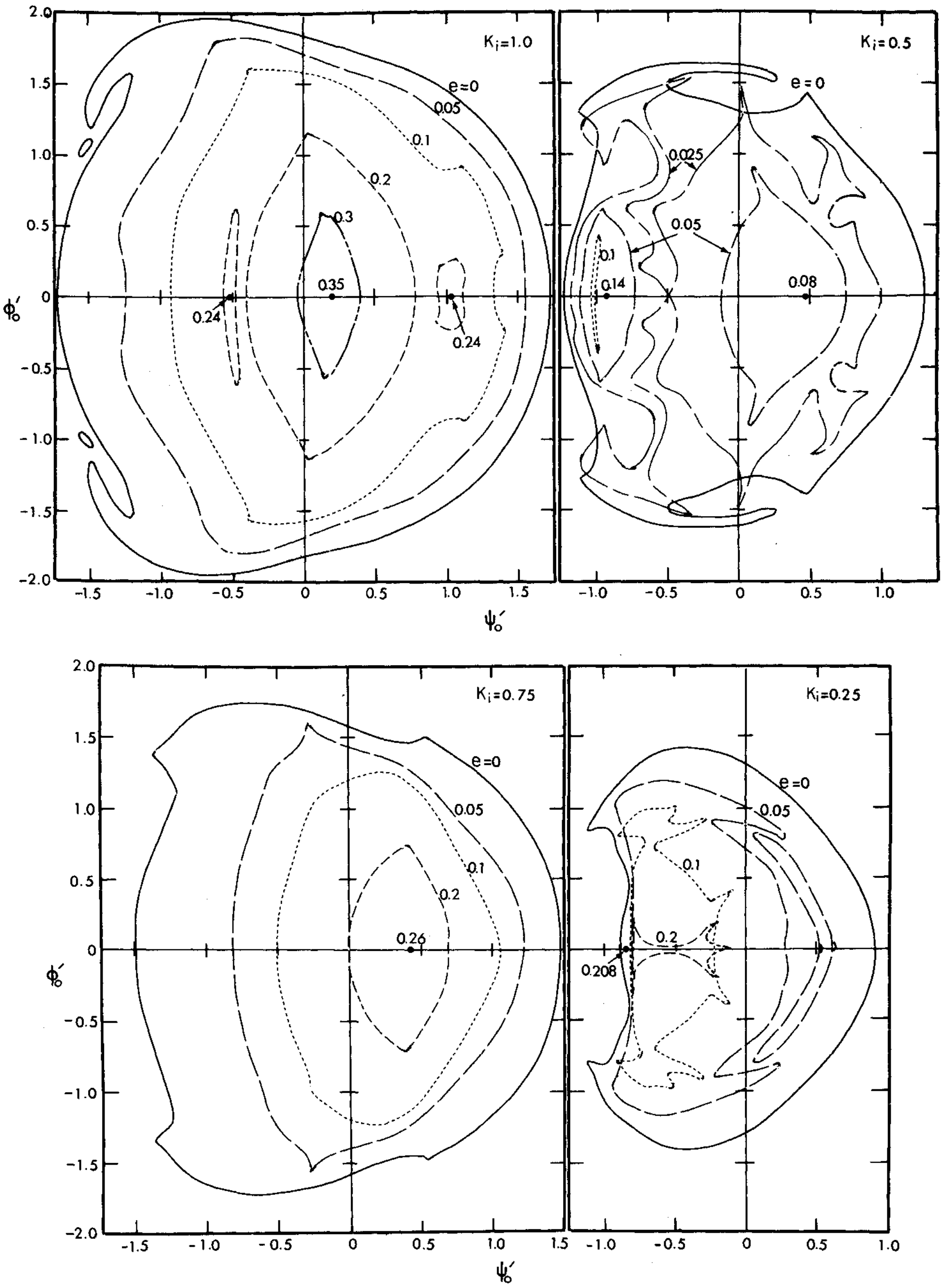
For autonomous systems, use of the concept of integral manifold in conjunction with the constant Hamiltonian facilitated the stability analysis appreciably.[8,10,12] Unfortunately, in presence of eccentricity, the concept loses its importance due to the obvious difficulty in representing and interpreting the hyper-surfaces in phase space. Hence an alternate approach is necessary to get meaningful information about the system stability.

Here the stability bounds are established by analyzing the librational response, over 15–20 orbits, to systematically varied initial conditions, satellite inertia, and orbit eccentricity. A variable secant approach and the symmetry properties of Eq. (5) reduced the computational effort substantially. The vast amount of information, thus gathered, is condensed in the form of design plots (Fig. 3), which indicate allowable impulsive disturbances $\psi_0, \phi_0 = 0$ at perigee for nontumbling motion, over a range of K_i and e. For comparison, earlier results with circular orbits are also included.[10]

The effect of even a slight increase in eccentricity is to rapidly reduce the stability region, particularly for satellites with smaller K_i. The reduction, in general, is more severe in the plane of the orbit, where the satellite is able to withstand a relatively large positive disturbance. The plots remain symmetrical about $\phi' = 0$ as in the case of the autonomous system. The peculiar shape of a stability region with numerous spikes, may be attributed to the predominance of various periodic solutions.[17,18] Of course, at the highest eccentricity for stable motion, the only available solution is a periodic one as indicated by dots in Fig. 3. The crossing of stability bounds suggest that in some situations, increase in eccentricity may be favorable, locally, in the system stabilization.

The reduction of K_i to 0.25, i.e., a value less than the critical $\frac{1}{3}$, reverses some of the trends established above. This is apparent from its better stability characteristics compared to $K_i = 0.5$ in eccentric orbits. A shorter satellite also exhibits an ability to withstand a larger negative impulse, $-\psi_0'$.

Although the plots presented here are for disturbances received at perigee, averaging over a large number of orbits suggests their applicability, at least approximately, to any θ in small eccentricity orbits. In principle, the system behaviour is similar to planar librations in an elliptic orbit[5] and coupled librations in a circular orbit.[8–10] It is important to



a) $K_i = 1.0, 0.5$

b) $K_i = 0.75, 0.25$

Fig. 3 Effect of satellite inertia and orbit eccentricity on the allowable impulsive disturbances for stable motion; $\theta_0 = \psi_0 = \phi_0 = 0$.

recognize that presence of the cross-plane motion improves the satellite's ability to withstand impulsive disturbances.

Effect of Aerodynamic Torque on System Response and Stability

Equation of Motion

Using Schaaf and Chambré's approach[19] for a satellite surface in a free molecular flow, the modified potential function for a cylindrical satellite in a circular orbit was given by Shrivastava and Modi.[12] In an elliptic orbit, the change in density and orbital velocity can be expressed as:

$$\rho = \rho_p[(r - R)/(r_p - R)]^n \quad (16)$$

$$v^2 = v_p^2(v/v_p)^2 = v_p^2[(2r_p - r + re)/(r + re)] \quad (17)$$

The value for the exponent n varies between -5 to -7 in the altitude range of 100–500 miles. The aerodynamic potential thus becomes:

$$U_a = I\dot{\theta}_p^2 B_{fp}\{[(r/r_p - R/r_p)/(1 - R/r_p)]^n(2r_p/r - 1 + e)/(1 + e)\}[\psi + S_\psi(C_\psi + C_1 S_\psi)]/2 \text{ (for } |\psi| \leqslant \pi/2) \quad (18)$$

where

$$B_{fp} = \rho_p C_D e D_0 L_0 v_p^2/2I\dot{\theta}_p^2 \quad (19)$$

$$C_1 = \pi D_0/4L_0 = \pi[(1 - K_i)/12(1 + K_i)]^{1/2} \quad (20)$$

It is apparent that the governing equation of motion in the ϕ degree of freedom remains unchanged and that in the ψ degree modifies to:

$$\psi''(1 + eC_\theta) - 2eS_\theta(\psi' + 1) - 2\phi'(\psi' + 1)(1 + eC_\theta)T_\phi + 3K_i S_\psi C_\psi + B_{fE}(1 + eC_\theta)C_\psi(|C_\psi| + C_1 S_\psi)/C_\phi^2 = 0 \quad (21)$$

where

$$B_{fE} = B_{fp}\{[(1 + e)/(1 + eC_\theta) - R/r_p]/(1 - R/r_p)\}^n(1 + 2eC_\theta + e^2)(1 + e)^2/(1 + eC_\theta)^4 \quad (22)$$

Note that the system remains invariant under the transforma-

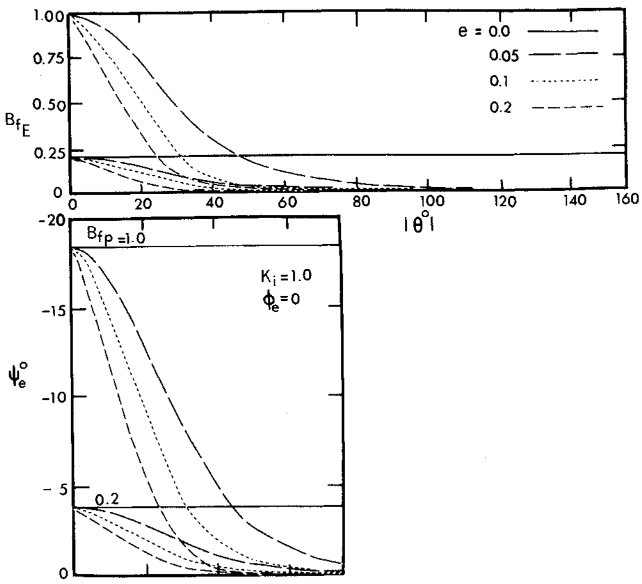


Fig. 4 Variation of aerodynamic coefficient and stable equilibrium configuration with θ and e.

tion θ,ψ,ϕ to $\theta,\psi,-\phi$. Increased complexity renders the analytical techniques of questionable value, particularly for motion in the large. Numerical methods, therefore, have to be resorted to.

Equilibrium Configuration and System Response

The stable equilibrium position is given as:

$$\psi_e \approx \tan^{-1}\{[-B_{fE}(1 + eC_\theta) + 2eS_\theta]/(3K_i + B_{fE}C_1)\} \tag{23}$$

As B_{fE} varies with θ, ψ_e changes continuously (Fig. 4). The symmetry of the plots about $\theta = 0$ is of interest. The presence of eccentricity tends to confine the effects of aerodynamic perturbations to the region near perigee. Even for small eccentricity orbits, $e < 0.1$, the aerodynamic torque becomes negligible for $|\theta| > 60°$. The rate of reduction enhances with increasing e and B_{fp}, and decreasing K_i.

A few representative response plots, obtained numerically, are shown in Fig. 5. As against the librational motion about a constant equilibrium position in a circular orbit, the presence of a forcing function along with the periodic variation of aerodynamic torque and equilibrium configuration makes the resulting response quite complex. The modulations, which are more predominant in the planar degree of freedom grow rapidly with B_{fp}. Even for an identical disturbance in the two degrees of freedom, the planar component appears to be more susceptible to instability. This, in a sense, justifies earlier simplified models of planar librations used by several authors.[1–3,5,17]

As can be expected, the forcing function arising from orbit eccentricity induces planar librational motion. Due to aerodynamic influence, planar oscillations were noticed in absence of any external disturbance, even in a circular orbit.[12] The combined effect of e and B_{fE} results in a considerably larger planar motion particularly for short satellites. The character of the response suggests possible reduction in the stability region due to an aerodynamic torque.

All the response data presented so far, correspond to stable operation of the satellite. Its critical dependence on satellite inertia and orbit eccentricity was shown through stability plots. In Figure 6 are shown several examples of instability as functions of K_i, B_f, and e. Note that a slender satellite, $K_i = 1.0$, moving in a circular orbit through a pure gravity gradient field, executes large amplitude stable librations when subjected to a unit impulse, $\psi_0' = \phi_0' = 1$; $\psi_0 = \phi_0 = 0$. However, change in system parameters beyond the critical value lead to a tumbling motion. For instance, the reduction of K_i to 0.5 or the increase of eccentricity to 0.15 lead to instability within a short time. An increase in B_f to a unit value initiates clockwise tumbling in a circular orbit itself. It may be pointed out that in all these cases, the motion across the orbit remains bounded. Importance of a parametric study of the system, from design considerations, is thus apparent.

Stability Plots

The stability of the system is established, as before, numerically. Design plots again prove useful in condensing the enormous amount of information (Fig. 7). The plots, symmetrical about $\phi_0' = 0$, include the corresponding results for a circular orbit as given in Ref. 12. It is apparent that even small eccentricity of the orbit makes the stability region shrink substantially. The presence of an aerodynamic torque further enhances this trend. As in the case of eccentricity,

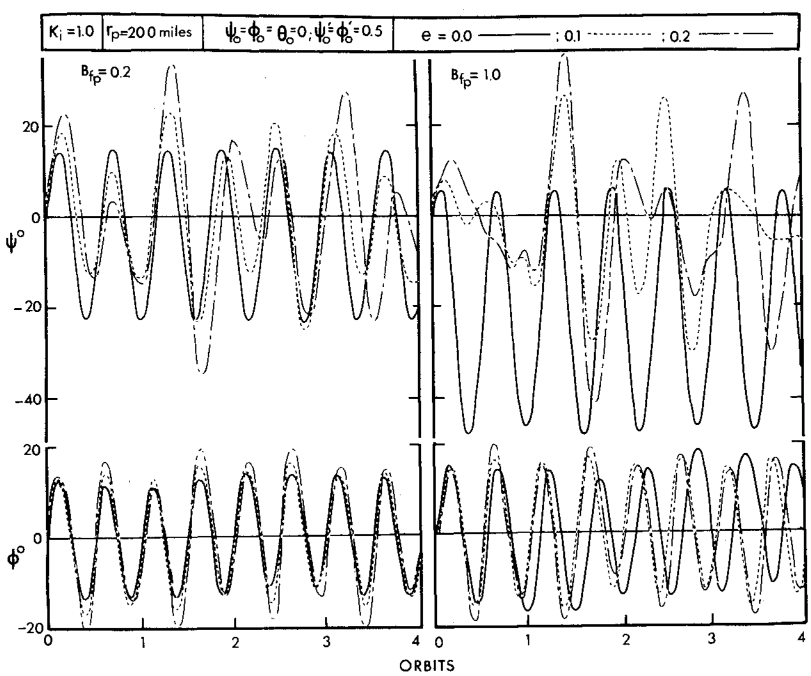


Fig. 5 Typical system response showing the effect of orbit eccentricity and atmospheric torque.

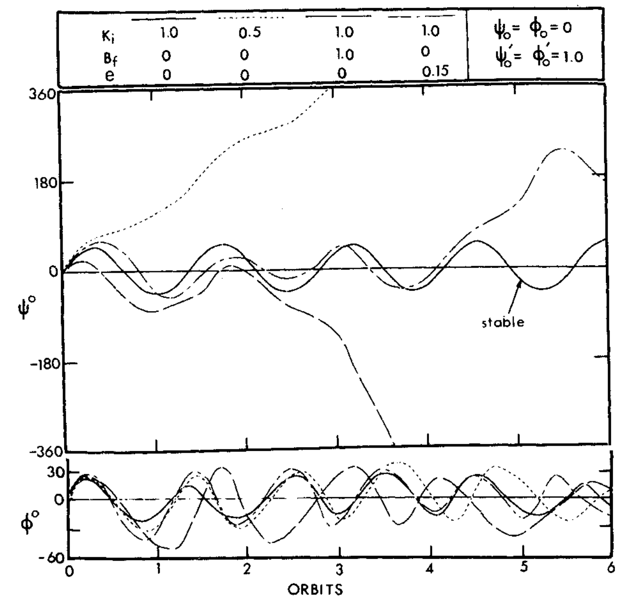


Fig. 6 Instability excited by change of system parameters.

the reduction in the stability margin is predominantly in the ψ degree of freedom.

The aerodynamic torque represents a periodic disturbance. Although active only over a relatively small portion of the satellite's eccentric orbit, it has considerably adverse influence on the stability.

Aerodynamic Damping

Use of environmental forces in librational damping and attitude control is not new. Paul et al.[20] showed the feasibility of a magnetic hysteresis damper interacting with the earth's magnetic field. The application of the solar radiation pressure for a space vehicle propulsion during interplanetary flights has been proposed by several authors including Garwin,[21] who described it as solar sailing. Sohn[22] et al. investigated specific configurations for satellite stabilization with respect to the sun. More directly, Mallach[23] suggested the use of solar radiation as a damping force for gravity oriented satellites. Recently, Modi et al.[24–25] established the feasibility of using solar radiation pressure for an efficient planar damping and attitude control by adjusting the exposed areas of solar pads as a function of librational velocity and angle.

This section explores the possibility of utilizing the normally destabilizing aerodynamic moment to advantage. A semi-passive, velocity-sensitive controller provides a restoring moment of appropriate magnitude and sense through judicious adjustment of flaps exposed to the free molecular flow. This concept of librational damping through differential lift is essentially an extension of the aircraft stabilization technique.

Feasibility of the Concept

Introduction of the aerodynamic force to a gravity gradient system presents a possibility of center of pressure not coinciding with the center of mass. This leads to an aerodynamic torque which, if controlled efficiently, can provide not only librational damping but also attitude control of the satellite. Extending the concept of aircraft attitude control to a spacecraft moving in the rarefied atmosphere, consider a satellite, with two identical stabilizing flaps, as shown in Fig. 8a. The flaps, located in the local horizontal plane passing through the center of mass of the satellite and controlled independently, are free to rotate about the axes perpendicular to the line of symmetry of the satellite. An equal and opposite rotation of the flaps, leads to a moment about the center of mass which has stabilizing components in both ψ and ϕ degrees of freedom. Thus with librating satellites, flap orientation can be adjusted continuously to provide a suitable correcting torque. The moment due to the forces being balanced, no rotation about the z axis (yaw) is induced. As the satellite under the action of various disturbances starts to librate, the flaps are inclined appropriately with respect to the impinging stream to provide a stabilizing torque. If this torque were controlled as a function of the satellite's librational velocity it should be able to damp the motion.

Figure 8b shows some of the alternate schemes for flap arrangement. Although the arrangement discussed previously is likely to be the simplest to construct, it has obvious limitations, e.g., lack of control in the individual degree of freedom. The triangular setting, Fig. (8b) (i), in which the front flap damps the planar motion while the rear two by their opposite movement control the cross-plane librations, provides a way of governing the individual degree of freedom. For maintaining the axisymmetric character, the rear flaps are appropriately off-set from the center line of the satellite. A further improvement, in terms of symmetry and magnitude of the restoring moment, is represented by the configuration shown in

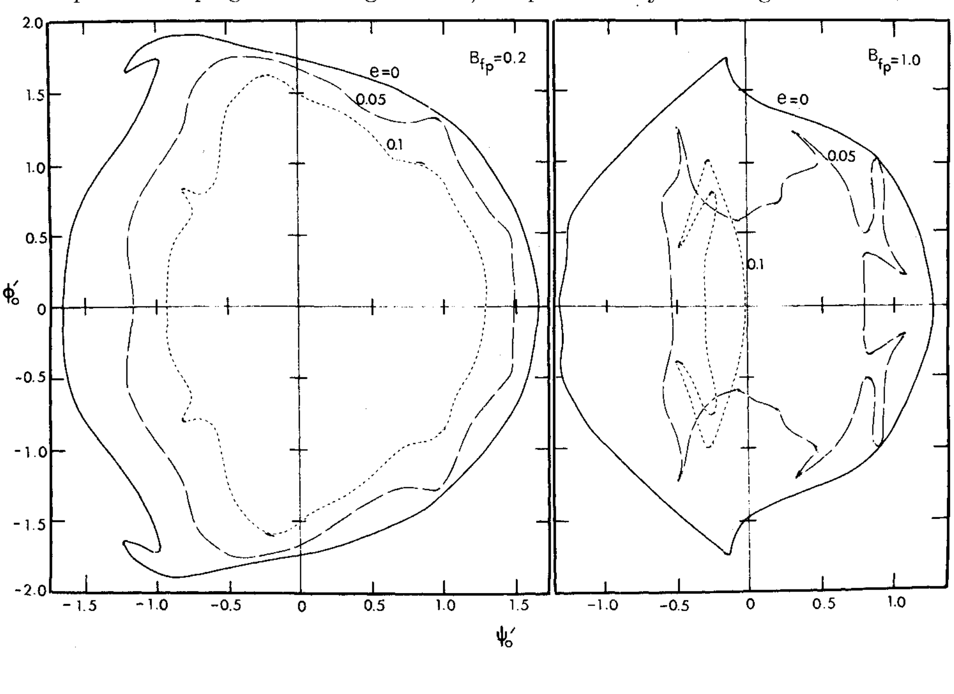
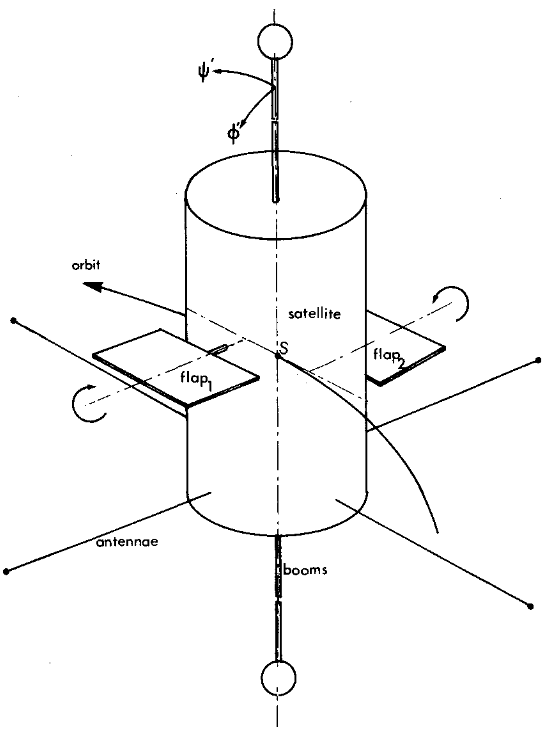
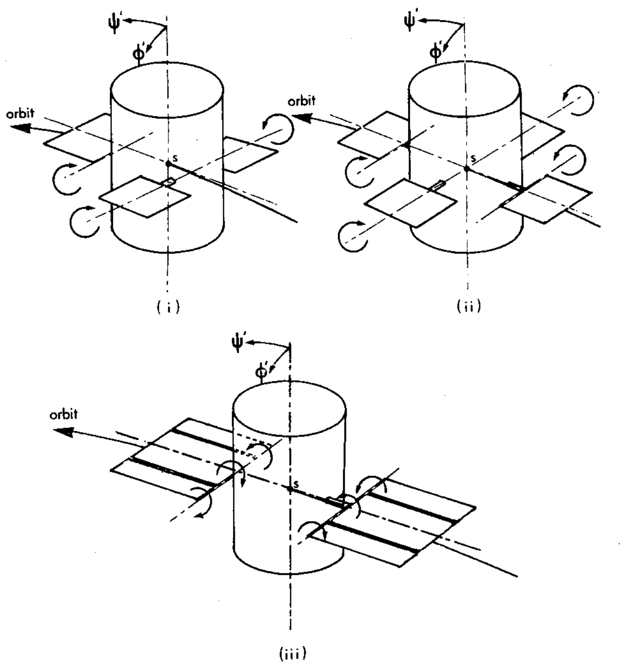


Fig. 7 Effect of aerodynamic torque and orbit eccentricity on the allowable impulsive disturbance for stable motion, $K_i = 1$, $\theta_0 = \phi_0 = 0$, $\psi_0 = \psi_e$.



a) Satellite configuration



b) Possible arrangements of stabilizers

Fig. 8 Aerodynamic damping and stabilization.

Fig. 8b (ii). Here the off-set is eliminated without affecting the independent control of each degree of freedom. Introduction of a set of split-flaps, Fig. 8b (iii), represents another possibility. Here the center sections of each assembly are actuated individually to provide a torque for planar control, while the outer flaps damp the motion in ϕ degree of freedom. Numerous other variations can be thought of by combining these basic arrangements.

The concept during actual design may be faced with several optimization problems: 1) the atmospheric density as well as the lift coefficient reduce rapidly with increase in altitude. On the other hand, the life time of the satellites diminishes with their closeness to the Earth. Thus a compromise is indicated. 2) The flaps should be light yet sufficiently rigid and large to generate enough lift. Furthermore, the drag should be small to minimize orbital perturbations. 3) The flaps should be so located as to avoid interference with the operation of antennae, cameras, solar cells, or scientific instruments. 4) The arms supporting the flaps should be long enough for adequate moment without sacrificing lightness and rigidity. Obviously extensive ground tests would be required.

Angular movement of well-arranged flaps is likely to have little effect on total inertia, axisymmetric character of the system or the position of the center of mass. Of course, sensing the disturbance and operation of the flaps may involve time delay. This, however, would be of little significance due to long period, order of orbital period, of the librations.

Damped Response

With linearly proportional, velocity-sensitive control, the governing equations of motion in a circular orbit modify to

$$\psi'' - 2\phi'(\psi' + 1)T_\phi + 3K_iS_\psi C_\psi + B_f(|C_\psi| + C_1S_\psi)C_\psi/C_\phi^2 + \mu_1\psi' = 0 \quad (24a)$$

$$\phi'' + [(\psi' + 1)^2 + 3K_iC_\psi^2]S_\phi C_\phi + \mu_2\phi' = 0 \quad (24b)$$

where μ_1 and μ_2 are positive proportionality constants and B_f is the constant aerodynamic coefficient for the satellite without flaps. Because of the axisymmetric arrangement, the stabilizers do not induce rotations about z axis. The foregoing ignores any variations in damping torque due to small λ oscillations caused by coupling effects [Eq. (3)].

The physical size and location of the flaps would also impose a limit on the stabilizing torque. The following condition implies that the flaps would maintain their orientation for torque requirement beyond $\tau_{i\max}$.

$$|\mu_1\psi'| \leqslant \tau_{1\max}, \quad |\mu_2\phi'| \leqslant \tau_{2\max} \quad (25)$$

where,

$$\tau_{i\max} = \rho v^2 A_{fi} C_{L_{\max}} l_{mi}/2I\dot{\theta}^2 \quad (26)$$

For example, a satellite, with $I = 600$ slug/ft², in a circular orbit at 200 miles alt, $\rho \simeq 3.0 \times 10^{-13}$ slug/ft³ – ARDC 1959 (Ref. 26) and provided with two 3 ft × 3 ft flaps with moment arm of 5 ft each, has maximum coefficient of lift equal to about 0.2 (Ref. 19) and the associated $\tau_{i\max}$ becomes 2.0.

Figure 9 shows, over 3 orbits, the effect of controller proportionality constants μ_1,μ_2 and system parameters on librational response. A slender satellite with a small aerodynamic coefficient B_f, undergoes substantially large motion in absence of

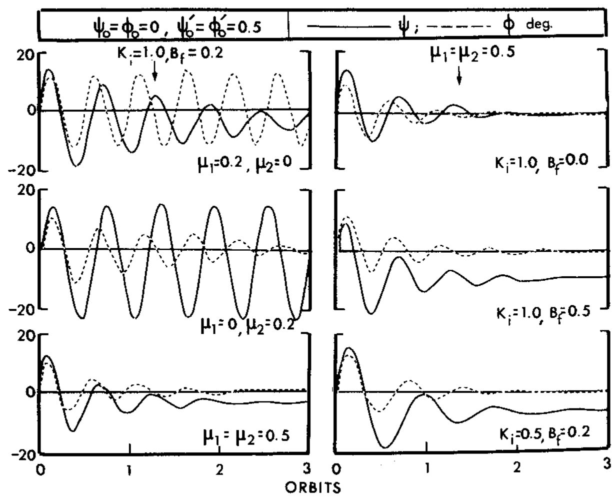


Fig. 9 Aerodynamically damped response ($\tau_{i\max} = 2$) in circular orbits showing the effects of system parameters.

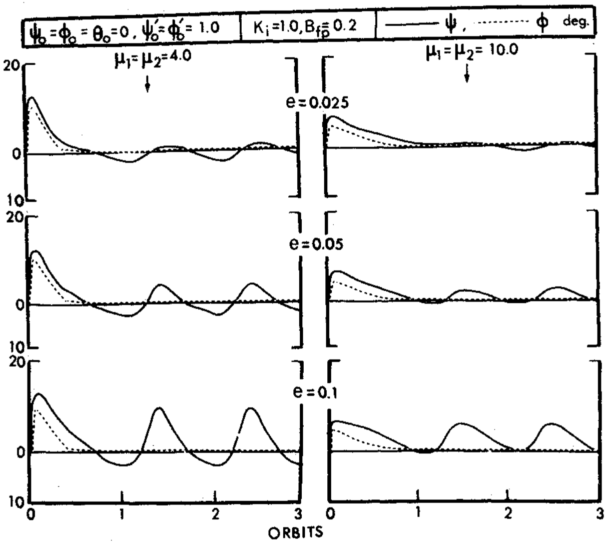


Fig. 10 Effectiveness of the aerodynamic controller in elliptic orbits; $(\tau_{i\max})_p = 2$, $r_p = 200$ miles.

damping. However, a small stabilizing torque in either ψ or ϕ direction causes a quick reduction in amplitude. Increase in μ_1,μ_2 considerably improves the damping efficiency. The time index, time to damp or to attain limit-cycle, may be as small as the orbital period.

The effectiveness of this aerodynamic damping concept with reference to satellites of different K_i and B_f is also suggested. It appears that irrespective of the transient response, which strongly depends on the system parameters, the time to damp remains relatively unaffected. Final configuration attained in each case is the stable equilibrium position, which depends on K_i and B_f only. The time index, of course, increases with the magnitude of a disturbance, yet even in the worse situation considered, $\psi_0' = \phi_0' = 2.0$, it is limited to three orbits. Bounded response to the normally destabilizing disturbances suggests improved stability.

The mechanism appears to be quite effective in librational damping of near-Earth satellites. Its efficiency in controlling general motion appears to be, at least, equal to that of conventional viscous dampers[27] and solar pressure stabilization[24,25] in planar motion.

It is interesting to note that if the aerodynamic moment, which involves several variable parameters, were adjusted appropriately so as to be a function of librational velocity as well as angular displacement, the mechanism could stabilize the satellite at any desired orientation. This would represent a simple yet powerful method of attitude control.

For the satellites in elliptic orbits, the maximum possible torque varies continuously with the satellite position and is given by:

$$\tau_{i\max} = (\tau_{i\max})_p B_{fE}/B_{fp} \tag{27}$$

The mechanism, although effective only over a small portion of the trajectory continues to damp the system [Eqs. (21) and (5b)] response successfully (Fig. 10). Even for a large disturbance, the steady-state motion corresponds to a small amplitude limit cycle in ψ. The crossplane motion is damped completely as before. The effect of eccentricity is to increase the amplitude of the limit cycle, however, the time index still remains less than the orbital period. Note that the pointing accuracy near perigee, normally the region of importance, continues to be quite good. With a suitable choice of proportionality constants the performance of the system near perigee can be further improved. On the other hand, the relatively large amplitude periodic librations near apogee can be used to advantage in specific missions such as an orbiting telescope scanning a portion of the sky.

Conclusions

The important aspects of the analysis and more significant conclusions may be summarized as follows: 1) a simple closed form solution as given by Butenin's variation of parameter method can be used effectively during preliminary design of a satellite. 2) The stability region diminishes rapidly with increase in eccentricity. The shrinkage is more significant in the planar degree of feedom and for shorter satellites (smaller K_i). 3) The presence of aerodynamic torque affects the stable equilibrium configuration. It changes periodically with the position of the satellite in an eccentric orbit. Atmospheric torque leads to rapid decrease in size of the stability region. 4) Aerodynamic damping of librational motion appears to be quite promising. A velocity-sensitive, semipassive controller can damp even large amplitude motion in less than two orbits.

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NOVEMBER 1971 AIAA JOURNAL VOL. 9, NO. 11

Local Modifications of Damped Linear Systems

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A procedure is developed for determining the eigenvalues and eigenvectors of a discrete linear-vibration system resulting from the addition or removal of a discrete element. In this procedure the known characteristics of the original system are used to generate the modified characteristic equation directly without having to solve the modified eigenvalue problem explicitly. Because of the form of the modified characteristic equation, the problem is ideally suited to numerical solution by the Newton Raphson iteration procedure. If repeated eigenvalues exist, the system matrices may not be diagonalizable by classical modal methods. However, the system can be reduced to Jordan Canonical form and the procedure presented incorporates this possibility.

Introduction

ANALYTICAL investigations of dynamic elastic systems frequently involve a determination of the effect of a change in a particular system component on the natural frequencies and normal modes of vibration. Often, a change of this type is not sufficiently small to permit a solution by standard perturbation techniques, and alternative procedures are required. In an investigation of some problems associated with the addition of localized springs and masses to undamped one-dimensional continuous systems, Weissenburger[1] developed a method of analysis called "eigenvalue modification" which is applicable for both small and large modifications. This procedure utilizes an eigenfunction expansion to express the eigenfunctions (normal modes) of the modified system in terms of the known eigenfunctions of the unmodified system; but since the actual computation is performed with only a finite number of terms, the problem is equivalent to an investigation of a local modification of a symmetric positive definite matrix eigenvalue problem. Although the use of an eigenfunction expansion to solve such problems is in no way unique, his method of solving for the frequencies and modes associated with such a modification is much simpler and more accurate than previous procedures.[2]

In this paper, Weissenburger's procedure is recast into matrix form for direct application to discrete vibration systems. It is also extended to include the effects of linear viscous damping, both in the original system and in the modification, with the result that the eigenvalues may be complex quantities, and the eigenvectors associated with multiple eigenvalues may not form a spanning set for the vector space. In addition, the restriction to symmetric sign-definite‡ matrices, which was an implicit requirement in Weissenburger's work with self-adjoint positive-definite systems, is relaxed in this development and the simplifications which arise as a result of distinct eigenvalues and symmetric positive-definite matrices are indicated.

Derivation of Equations

Original System

In the solution of the modified eigenvalue problem arising from a localized change in an n degree of freedom vibration system, a complete knowledge of the original system eigenvalues and eigenvectors is required. With the inclusion of linear-viscous damping, these original characteristics are assumed to be those associated with the equivalent $2n$ dimensional matrix system:

$$\left(\mu \begin{bmatrix} [0] & [m] \\ [m] & [c] \end{bmatrix} + \begin{bmatrix} -[m] & [0] \\ [0] & [k] \end{bmatrix}\right) \begin{Bmatrix} \mu \{X\} \\ \{X\} \end{Bmatrix} = \{0\} \quad (1)$$

Presented as Paper 71-215 at the AIAA 9th Aerospace Sciences Meeting, New York, January 25–27, 1971; submitted February 16, 1971; revision received July 2, 1971. This work was supported by NASA.

Index category: Structural Dynamic Analysis.

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‡ In this paper, the term sign-definite refers to either positive definite or non-negative definite.